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Notes on the dynamics of noncommutative lumps

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Abstract

We consider a pair of noncommutative lumps in the noncommutative Yang–Mills–Higgs model, which is a perturbative branch of the infinite-dimensional BFSS M(atrix) model. In the case when the lumps are separated by a finite distance their 'polarizations' do not belong to orthogonal subspaces of the Hilbert space. In this case the interaction between lumps is nontrivial. We analyse the dynamics arising due to this interaction both in a naive approach of rigid lumps and exactly as described by the underlying gauge model. It appears that the exact description is given in terms of finite-matrix models/multidimensional mechanics whose dimensionality depends on the initial conditions.

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1. Introduction

Recent progress in theories over noncommutative spaces (for a review see e.g. [1–4] and references therein) is stimulated by their importance for the nonperturbative dynamics of string theory [5–8].

One particular feature of noncommutative field theories which has attracted considerable interest is that in noncommutative models there exists a class of localized solutions nonexistent in the models on commutative spaces. Although such solutions are different from a *soliton* in the usual sense, they are conventionally called 'noncommutative solitons'. In this work we consider a subclass of such configurations. As it appears that the 'noncommutative solitons' which we deal with in this work have zero energy at rest; a more adequate term for them would be either 'vacuum' or 'lump'. Throughout this paper we use the latter.

Noncommutative lumps, in a scalar model with a potential having nontrivial local minima, were first discovered in the limit of strong noncommutativity [9]. These solutions were interpreted as condensed low-dimensional branes living on a higher-dimensional noncommutative brane [10, 11]. Their generalization was found in the case of an arbitrary noncommutativity parameter by allowing a nontrivial gauge field background [12–15]. These solutions correspond to nontrivial gauge field fluxes [16–18]. The particular property of lump solutions we are considering is that they are 'made' of gauge fields only. However, using the equivalence between different noncommutative Yang–Mills–Higgs models [19, 20], these configurations can be mapped into noncommutative solitons in the sense of [14] or others.

The general multi-lump solutions appear like sums of projectors to mutually orthogonal finite-dimensional subspaces of the Hilbert space. If subspaces are not orthogonal the configuration fails to be a static one. Therefore the lumps start to interact.

An approach to the description of interacting lumps was proposed in [21, 22]. This approach uses the substitution of the configuration consisting of a pair of lumps by a close one which belongs to the static solutions. The interaction of the lumps in the adiabatic approach is described by the motion in the curved modulus space of static solutions. This approach, however, would only be valid provided that the motion was slow and stable in the vicinity of the modulus space. In our case there are, however, indications that the noncommutative lumps are not stable dynamically (compare with [23]), which also leads to the instability of the motion around the modulus space.

Our approach is free from these drawbacks since we do not make any assumptions about stability and adiabaticity. As the analysis shows, the dynamics of the system does not appear to be a stable one. Moreover, it is stochastic. The regular motion occurs only when the distance between lumps is exactly $\sqrt{\theta \ln 2}$. Even a small deviation from this brings the system to the stochastical regime. It is also interesting to note that for some natural initial conditions the dynamics of noncommutative lumps is described by a finite-dimensional matrix model.

The plan of this paper is as follows. First, we introduce the reader to the noncommutative lumps in the Yang–Mills–Higgs model. After that we analyse the lump dynamics in both the naive approach, when we treat the lumps as rigid particles and neglect the dynamics of their 'shapes', and an exact approach, when all possible deformations are taken into account. The comparison reveals unexpected features in the behaviour of the interacting lumps. Namely, their real behaviour is completely different from naive expectations.

2. The model

We consider the noncommutative gauge model which is described by the following action:

$$S = \int \mathrm{d}t\, \mathrm{tr}\left(\frac{1}{2}\dot{X}^i\dot{X}^i + \frac{1}{4g^2}[X^i, X^j]^2\right), \tag{1}$$

where fields $X^i, i = 1, \ldots, D$, are time-dependent Hermitian operators, acting on Hilbert space $\mathcal{H}$, which realizes a irreducible representation for the one-dimensional Heisenberg algebra generated by

$$[x^1, x^2] = \mathrm{i}\theta. \tag{2}$$

Operators x^μ satisfying the algebra (2) are said to be the coordinates of a noncommutative two-dimensional plane. In this interpretation the operators of the Heisenberg algebra $\mathcal{H}$ can be represented through ordinary functions given by their Weyl symbols. The composition rule for the symbols is given by the Moyal or star product,

$$f * g(x) = \mathrm{e}^{\mathrm{i}(\theta/2)\epsilon^{\mu\nu}\partial_\mu\partial'_\nu} f(x)g(x')\big|_{x'=x}, \tag{3}$$

where $f(x)$ and $g(x)$ are Weyl symbols of some operators, $f * g(x)$ is the Weyl symbol of their product and ∂_μ, ∂'_μ denote derivatives with respect to x^μ and x'^μ, respectively.

The integral of a Weyl symbol corresponds to $2\pi\theta\times$ the trace of the respective operator, while its partial derivative derivative with respect to x^μ corresponds to the commutator,

$$\partial_\mu f(x) = \mathrm{i}(p_\mu * f - f * p_\mu)(x) = [p_\mu, f](x), \tag{4}$$

where p_μ is given by $p_\mu = (1/\theta)\epsilon_{\mu\nu}x^\nu$. Since there is one-to-one correspondence between operators and their Weyl symbols we shall not distinguish between them, i.e. we shall retain the same character for both, unless in danger of confusion.

The model (1) corresponds to the Hilbert space ($N \to \infty$) limit of the BFSS matrix model as well as in different perturbative limits it describes the noncommutative Yang–Mills(–Higgs) model in the temporal gauge[1] $A_0 = 0$ [19, 20, 24].

Indeed, for equations of motion corresponding to the action (1),

$$\ddot{X}_i + \frac{1}{g^2}[X_i, [X_i, X_j]] = 0, \tag{5}$$

one may find the static classical solution $X_i = p_i$ [19, 20, 24], satisfying

$$[p_i, p_j] = \mathrm{i}\theta^{-1}_{ij}, \tag{6}$$

with constant invertible θ^{-1}_{ij}. We also assume for the solution the irreducibility condition. Namely, the set of operators p_i is such that for any operator F, from $[p_i, F] = 0$ with all p_i, $i = 1, \ldots, D$, it follows that F is a c-number, $F \sim \mathbb{I}$.

Expanding fields around this solution, $X_i = p_i + A_i$, and Weyl ordering operators A_i with respect to $x^i = \theta^{ij}p_j$, one obtains precisely the $(D + 1)$-dimensional noncommutative Yang–Mills model for the field given by the Weyl symbol $A_i(x)$.

Obtaining another solution with a smaller number of independent p_i, $X_\alpha = p_\alpha$, $\alpha = 1, \ldots, p$,

$$[p_\alpha, p_\beta] = \mathrm{i}\theta^{-1}_{\alpha\beta}, \tag{7}$$

and $X_i = \text{const}$, $i = p + 1, p + 2, \ldots, D = 0$, one obtains as a result the model of a p-dimensional Yang–Mills field interacting with $(D - p)$ scalars.

Having in mind this equivalence, in what follows we shall consider the two-dimensional form of this noncommutative model. Also neglecting for a while the issues connected with the Gauss law, the theory appears like a noncommutative *scalar* model in $(2 + 1)$ dimensions.

For our purposes it is convenient to use two-dimensional 'complex coordinates' given by oscillator raising and lowering operators[2] a and $\bar{a}$,

$$a = \frac{1}{\sqrt{2\theta}}(x^1 + \mathrm{i}x^2), \qquad \bar{a} = \frac{1}{\sqrt{2\theta}}(x^1 - \mathrm{i}x^2), \qquad [a, \bar{a}] = 1, \tag{8}$$

and the oscillator basis,

$$\bar{a}a|n\rangle = n|n\rangle, \qquad a|n\rangle = \sqrt{n}|n-1\rangle, \qquad \bar{a}|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{9}$$

As one can see the solution (6) or (7) has divergent traces. Another type of static solution that one can find in model (1) is given by a configuration with localized, i.e. lumplike, Weyl symbols (in some background p_i)[3]. It is given by commutative matrices of finite ranks [14]. Since these lumps carry no energy—they are geometrically nontrivial vacua—we shall call

[1] In this case the Gauss law constraints should be satisfied as well. We postpone the discussion of the Gauss law constraints until section 3.4.

[2] For the Weyl symbols we shall use later z and $\bar{z}$ instead of a and $\bar{a}$ to distinguish them from the Hilbert space operators.

[3] Fairly speaking these solutions are localized if the fields are treated as scalar ones. Since the gauge field definition $A_\alpha = X_\alpha - p_\alpha$ implies subtraction of a linear function p_α this type of solution corresponds to functions with linear growth.

them noncommutative lumps in spite of their close relation to 'noncommutative solitons' discussed in the literature [9–15].

Up to a gauge transformation the N-lump solution is given by

$$X^i = \sum_{n=0}^{N} c_n^i |n\rangle\langle n|, \tag{10}$$

where c_n^i is the nth eigenvalue of the (finite-rank) operator X^i. Due to the finiteness of the rank the Weyl symbol of X^i vanishes at infinity as quickly as a Gaussian factor multiplied by a polynomial. The simplest one-lump solution can be written in the form

$$X_i^{(0)} = c_i|\psi\rangle\langle\psi|, \tag{11}$$

where c_i give the 'height' and the 'orientation' of the lump. By a proper Lorentz transformation, $X_i \to \Lambda_i^j X_j$, one can make c_i have the only non-zero component, say c_1, while up to a gauge transformation the 'polarization' can be chosen to correspond to the oscillator vacuum state $|\psi\rangle = |0\rangle$.

In the star-product form operator (11) is represented by the Weyl symbol

$$X_i(\bar{z}, z) = 2c_i \mathrm{e}^{-2|z|^2}.$$

The lump shifted along the noncommutative plane by a (c-number) vector u is given by

$$X_i^{(u)} = c_i \mathrm{e}^{-\mathrm{i}p_\mu u^\mu}|0\rangle\langle 0|\mathrm{e}^{\mathrm{i}p_\mu u^\mu} = c_i \mathrm{e}^{-|u|^2}\mathrm{e}^{\bar{a}u}|0\rangle\langle 0|\mathrm{e}^{-a\bar{u}}. \tag{12}$$

Its Weyl symbol, correspondingly, is given by $X_i^{(u)}(z) = 2c_i\mathrm{e}^{-|z-u|^2}$. The shifted lump with constant u is a solution again. When u becomes time dependent one can perform the time-dependent gauge transformation to obtain[4]

$$X_i \to \mathrm{e}^{\mathrm{i}p_\mu u^\mu} X_i \mathrm{e}^{-\mathrm{i}p_\mu u^\mu}, \tag{13}$$

which shifts the lump back to the centre, but produce a kinetic term for $\sim\dot{u}^2/2$. Thus a single noncommutative lump moves freely like a *non-relativistic* particle. It is also stable since its energy at rest is zero.

In what follows we are going to analyse the situation when there is a pair of lumps separated by a distance u.

3. A pair of interacting lumps

As we have shown in the previous section, a single noncommutative lump can be always rotated to have the polarization $|0\rangle$ and orientation along X_1. In the case of two lumps one can choose *without loss of generality* for the configuration to involve nontrivially only two matrices e.g. X_1 and X_2.

Consider two lumps which are obtained from $c|0\rangle\langle 0|$ by shifts along the noncommutative plane by $u_{(1)}$ and $u_{(2)}$ respectively. The dynamics of the centre is free and can be decoupled by a time-dependent gauge transformation similar to (13).

Thus, the configuration we consider appears like

$$X_1 = cVPV^{-1} \equiv c|-u/2\rangle\langle -u/2|, \tag{14a}$$

$$X_2 = cV^{-1}PV \equiv c|u/2\rangle\langle u/2|, \tag{14b}$$

$$X_i = \text{const}, \qquad i = 3, \ldots, D, \tag{14c}$$

[4] This affects the gauge, $A_0 = 0$.

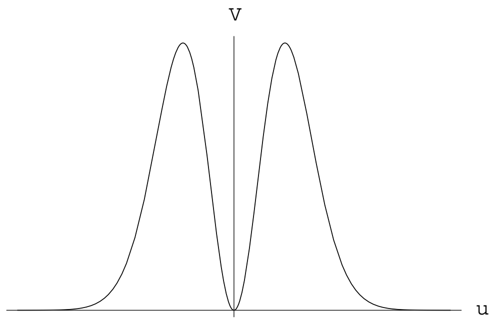

Figure 1. The profile of the lump–lump interaction potential in the naive approach.

where we introduced the shorthand notations

$$V = \mathrm{e}^{(\mathrm{i}/2)p_\mu u^\mu} = \mathrm{e}^{\frac{1}{2}(a\bar{u}-\bar{a}u)}, \tag{15}$$

$$P = |0\rangle\langle 0|. \tag{16}$$

The quotient c can be absorbed by the rescaling of the coupling and the time, therefore we can set it generically to unity, $c = 1$. This configuration corresponds to two lumps of the same height c oriented along X_1 and X_2, respectively, and separated by the distance u.

In what follows $X_i, i = 3, \ldots, D$, will also enter the equations trivially, so in the remaining part of the paper for the simplicity of notation the index i will run the range $i = 1, 2$. If we were considering more than two noncommutative lumps we would have to retain more matrices.

3.1. Naive picture: rigid lumps

Consider first a naive approach where we are dealing with rigid interacting lumps, which means that we are neglecting the deformations in their shapes. In this case the only parameter which is dynamical is the separation distance u. Although this approximation sounds reasonable, later we shall consider the exact description, which shows that this approach is not justified. However, we decided to retain this naive analysis for illustrative purposes.

To obtain the action describing the dynamics let us insert the ansatz (14) into the classical action (1). The computation of derivatives and traces gives the following u-dependence of the action:

$$S[u] = \int \mathrm{d}t \left(\frac{1}{2\theta}|\dot{u}|^2 - 2\mathrm{e}^{-\frac{|u|^2}{2\theta}} \left(1 - \mathrm{e}^{-\frac{|u|^2}{2\theta}}\right)\right), \tag{17}$$

where we restored the explicit θ dependence.

The potential is depicted in figure 1. According to this sufficiently close lumps attract while distant ones repel. At the critical distance $u_c = \sqrt{2\theta \ln 2}$ they will remain in unstable equilibrium.

The above conclusions concerning the lump dynamics would be valid, however, only in the case when one can neglect the involvement of the lump shape in the dynamics. To evaluate the importance of the shape dynamics one should consider arbitrary deformations of the shape of lumps and separate them from the motion of the lump as a whole.

In the next section we analyse the dynamics from the point of view of exact field equations of motion. The lump configuration is taken to be the initial condition for the field equations. The result we obtain in the next section will invalidate the results of the present naive approach; however, the critical distance u_c will correspond to a special case.

3.2. Exact description: lumps at rest

The exact description of the lump dynamics is given by the field equations of motion for X_i,

$$\ddot{X}_i + \frac{1}{g^2}[X_j, [X_j, X_i]] = 0 \tag{18}$$

corresponding to the action (1), supplied with initial conditions given by the lump background (14). Since the equations are second-order ones, in addition to this one has to consider the initial data for the time derivatives of X_i.

The simplest choice is when one starts at $t = 0$ with a static configuration. Thus, the initial conditions we impose are as follows:

$$X_1|_{t=0} = |-u/2\rangle\langle -u/2|, \qquad \dot{X}_1|_{t=0} = 0, \tag{19a}$$

$$X_2|_{t=0} = |u/2\rangle\langle u/2|, \qquad \dot{X}_2|_{t=0} = 0. \tag{19b}$$

Considering the lumps at the initial moment as being at rest produces a considerable simplification of the equations of motion. Indeed, the initial data (14) imply that the operators X_i are nonzero only on the two-dimensional subspace $\mathcal{H}_u$ of the Hilbert space which is the linear span of vectors $|u/2\rangle$ and $|-u/2\rangle$. Since, by virtue of equations of motion (18), the second time derivative is proportional to commutators of X_i, it also vanishes outside the two-dimensional subspace $\mathcal{H}_u$. Due to the zero initial conditions for the first derivatives, operators X_i will remain all the time in the same two-dimensional subspace of the Hilbert space.

Let us consider only those components of X_i which are nonzero. This reduces the Hilbert space operators to ones acting on the two-dimensional subspace $\mathcal{H}_u$ of the Hilbert space spanned by $|\pm u/2\rangle$. Let us introduce an orthonormal basis in $\mathcal{H}_u$.

The natural orthonormal basis which can be built up out of $|\pm u/2\rangle$ is given by vectors $|\pm\rangle$, defined as follows (see the appendix):

$$|+\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2(1+\mathrm{e}^{-\frac{1}{2}|u|^2})}}(|u/2\rangle + |-u/2\rangle), \tag{20a}$$

$$|-\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2(1-\mathrm{e}^{-\frac{1}{2}|u|^2})}}(|u/2\rangle - |-u/2\rangle). \tag{20b}$$

The singularity in $|-\rangle$ in the limit $u \to 0$ appears since in this limit $|u/2\rangle$ and $|-u/2\rangle$ tends to be parallel and the subspace become one dimensional.

In this basis our problem is reformulated in terms of the 2×2 matrix model with equations of motion superficially appearing the same as (18),

$$\ddot{X}_i^{(2)} + \frac{1}{g^2}[X_k^{(2)}, [X_k^{(2)}, X_i^{(2)}]] = 0, \tag{21}$$

but now $X_i^{(2)}$ are finite-dimensional 2×2 matrices. The initial conditions in the basis (20) are rewritten as follows:

$$X_1^{(2)}|_{t=0} = \tfrac{1}{2}\begin{pmatrix} 1+\mathrm{e}^{-\frac{1}{2}|u|^2} & -\sqrt{1-\mathrm{e}^{-|u|^2}} \\ -\sqrt{1-\mathrm{e}^{-|u|^2}} & 1-\mathrm{e}^{-\frac{1}{2}|u|^2} \end{pmatrix}, \qquad \dot{X}_1|_{t=0} = 0, \tag{22a}$$

$$X_2^{(2)}|_{t=0} = \tfrac{1}{2}\begin{pmatrix} 1+\mathrm{e}^{-\frac{1}{2}|u|^2} & \sqrt{1-\mathrm{e}^{-|u|^2}} \\ \sqrt{1-\mathrm{e}^{-|u|^2}} & 1-\mathrm{e}^{-\frac{1}{2}|u|^2} \end{pmatrix}, \qquad \dot{X}_2|_{t=0} = 0. \tag{22b}$$

It is worthwhile to note that the description in terms of 2×2 matrices is valid only for the situation where the lumps were initially at rest. Beyond this the condition $\dot{X}_i|_{t=0} = 0$ is an additional specification and this says also that the shapes of the lumps are not changing at the initial moment. One can consider a more general initial condition $\dot{X}_i|_{t=0} \propto X_i$, for which the same description in terms of 2×2 matrices remains valid. (One-soliton solutions of this type were considered in [25].)

Equations similar to ones given by (21) although in a different context, the one-dimensional ordinary Yang–Mills model, were under study for a long time and were initiated by [26–28]. In a modern context they appear in [29–32] in connection to the finite N matrix model. The system described by such equations was shown to exhibit a stochastic behaviour. Let us describe it in more detail in the application to the present case.

In order to rewrite the equations (22) in the scalar form let us expand the matrices X_i in terms of the two-dimensional Pauli matrices σ_α, $\alpha = 1, 2, 3$, and the two-dimensional unit matrix $\mathbb{I}_2$ (which in fact is the projector to $\mathcal{H}_u$) satisfying the algebra

$$[\sigma_\alpha, \sigma_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma}\sigma_\gamma, \qquad [\sigma_\alpha, \mathbb{I}_2] = 0. \tag{23}$$

The expansion is as follows:

$$X_{1,2} = X^0_{1,2}\mathbb{I}_2 + X^\alpha_{1,2}\sigma_\alpha. \tag{24}$$

In terms of this expansion the equations of motion appear as follows:

$$\ddot{X}^0_{1,2} = 0, \tag{25a}$$

$$\ddot{X}^\alpha_1 + \frac{1}{g^2}\left(X^2_2\delta^\alpha_\beta - X^\alpha_2 X_{2\beta}\right)X^\beta_1 = 0, \tag{25b}$$

$$\ddot{X}^\alpha_2 + \frac{1}{g^2}\left(X^2_1\delta^\alpha_\beta - X^\alpha_1 X_{1\beta}\right)X^\beta_2 = 0, \tag{25c}$$

where $X^2_{1,2} = X^\alpha_{1,2}X^\alpha_{1,2}$. For the initial conditions one also has

$$\dot{X}^\alpha_{1,2} = 0, \tag{25d}$$

$$X^0_1|_{t=0} = X^0_2|_{t=0} = \tfrac{1}{2}, \tag{25e}$$

$$X^1_1|_{t=0} = -\tfrac{1}{2}\sqrt{1 - \mathrm{e}^{-|u|^2}}, \qquad X^1_2|_{t=0} = \tfrac{1}{2}\sqrt{1 - \mathrm{e}^{-|u|^2}}, \tag{25f}$$

$$X^2_1|_{t=0} = X^2_2|_{t=0} = 0, \tag{25g}$$

$$X^3_1|_{t=0} = X^3_2|_{t=0} = \mathrm{e}^{-\frac{1}{2}|u|^2}. \tag{25h}$$

In particular, equation (25a) says that the scalar parts of the matrices $X_{1,2}$ remains constant during the motion ($X^0_{1,2}(t) = 1/2$) provided zero initial conditions for the 'velocities' $\dot{X}^0_i = 0$. At the same time the remaining parts are subject to more complicated nonlinear dynamics.

Before analysing the solutions for $X^\alpha_{1,2}$, $\alpha = 1, 2, 3$, let us consider their interpretation in terms of the lump dynamics over noncommutative space in the star-product representation.

The noncommutative function which corresponds to a particular solution $X^\alpha_i(t)$ will be given by

$$X_i(t; z, \bar{z}) = \tfrac{1}{2}\mathbb{I}(z, \bar{z}) + X^\alpha_i(t)\sigma_\alpha(z, \bar{z}), \tag{26}$$

where $X^\alpha_i(t)$ are the solutions of to (25) and $\mathbb{I}(z, \bar{z})$ with $\sigma_\alpha(x)$ the Weyl symbols corresponding to the Pauli matrices.

The respective two-dimensional Weyl symbols are computed in the appendix. They are given by

$$\sigma_1(z,\bar z) = \frac{2}{\sqrt{1-\mathrm{e}^{-|u|^2}}}(\mathrm{e}^{-2|z-\frac{u}{2}|^2} - \mathrm{e}^{-2|z+\frac{u}{2}|^2}), \tag{27a}$$

$$\sigma_2(z,\bar z) = \frac{2\mathrm{i}\mathrm{e}^{-2\bar z z}}{\sqrt{1-\mathrm{e}^{-|u|^2}}}(\mathrm{e}^{\bar z u-\bar u z} - \mathrm{e}^{-\bar z u+\bar u z}), \tag{27b}$$

$$\sigma_3(z,\bar z) = -\frac{2\mathrm{e}^{-\frac{1}{2}|u|^2}}{1-\mathrm{e}^{-|u|^2}}(\mathrm{e}^{-2|z-\frac{u}{2}|^2} + \mathrm{e}^{-2|z+\frac{u}{2}|^2}) + \frac{\mathrm{e}^{-2\bar z z}}{1-\mathrm{e}^{-|u|^2}}(\mathrm{e}^{\bar z u-\bar u z} + \mathrm{e}^{-\bar z u+\bar u z}), \tag{27c}$$

$$\mathbb{I}(\bar z, z) = \sigma_0(z,\bar z) = \frac{2}{1-\mathrm{e}^{-|u|^2}}(\mathrm{e}^{-2|z-\frac{u}{2}|^2} + \mathrm{e}^{-2|z+\frac{u}{2}|^2}) - \frac{2\mathrm{e}^{-2|z|^2-\frac{1}{2}|u|^2}}{1-\mathrm{e}^{-|u|^2}}(\mathrm{e}^{\bar z u-\bar u z} + \mathrm{e}^{-\bar z u+\bar u z}). \tag{27d}$$

As can be seen from equations (26) and (27), fields $X_i(z,\bar z)$ are different from zero only in the small vicinities (of size of the order of $\sim\sqrt{\theta}$) of points $z=0$ and $\pm u/2$. This property holds independently of the particular form of the solution $X_i^\alpha(t)$. This means that for any initial distance the lumps, once left with zero initial velocities, will not try to leave their places; the dynamics instead will concern only the heights and creation of a 'baby lump' at the mid-point between them. This behaviour is surprising as it is in total disagreement with the naive approach drawn in the previous subsection. There is no regime when the lumps would behave like rigid particles.

Let us consider now time-dependent functions $X_i^\alpha(t)$ in more detail. The equations (25) are too complicated to find the general solution; however, for our particular initial data one can use the rich symmetry of the model and find a simplifying ansatz.

Assuming that the magnitudes of X_1^α and X_2^α are also equal $X_1^2(t) = X_2^2(t)$ for nonzero times (we can check this assumption later as a consistency condition for the ansatz, but also prove it independently of the ansatz using conservation laws), one can split X_1^α and X_2^α into two orthogonal components X^α and Y^α as follows:

$$X_1^\alpha = X^\alpha + Y^\alpha, \qquad X_2^\alpha = X^\alpha - Y^\alpha, \tag{28a}$$

$$X^\alpha = \tfrac{1}{2}(X_1^\alpha + X_2^\alpha), \qquad Y^\alpha = \tfrac{1}{2}(X_1^\alpha - X_2^\alpha); \tag{28b}$$

the equality of the square modules $X_1^2(t) = X_2^2(t)$ implies that X^α and Y^α remain orthogonal. The equations of motion in terms of X^α and Y^α read

$$\ddot X^\alpha = -\frac{2}{g^2}Y^2X^\alpha, \tag{29a}$$

$$\ddot Y^\alpha = -\frac{2}{g^2}X^2Y^\alpha, \tag{29b}$$

where $X^2 = X^\alpha X^\alpha$ and $Y^2 = Y^\alpha Y^\alpha$. The initial conditions are respectively

$$X^\alpha|_{t=0} = \tfrac{1}{2}(X_1^\alpha(0) + X_2^\alpha(0)), \qquad Y^\alpha|_{t=0} = \tfrac{1}{2}(X_1^\alpha(0) - X_2^\alpha(0)), \tag{29c}$$

$$\dot X^\alpha(0) = \dot Y^\alpha(0) = 0. \tag{29d}$$

From equations (29) one can see that the directions of X^α and Y^α do not change. The fact that X^α and Y^α are always mutually orthogonal makes the assumption $X_1^2(t) = X_2^2(t)$ for the ansatz (28) consistent.

Splitting the vectors X^α and Y^α into the magnitudes X and Y, which are dynamical, and unimodular vectors

$$e_X^\alpha = X^\alpha/\sqrt{X^2}|_{t=0} = (0,0,1), \tag{30a}$$

$$e_Y^\alpha = Y^\alpha/\sqrt{Y^2}|_{t=0} = (1,0,0), \tag{30b}$$

which are conserved, one has the equations for the magnitudes X and Y,

$$\ddot{X} = -\frac{2}{g^2} Y^2 X, \tag{31a}$$

$$\ddot{Y} = -\frac{2}{g^2} X^2 Y. \tag{31b}$$

These equations are supplied by the initial data,

$$X|_{t=0} = \mathrm{e}^{-\frac{1}{2}|u|^2}, \qquad Y|_{t=0} = -\sqrt{1-\mathrm{e}^{-|u|^2}}, \qquad \dot{X}|_{t=0} = \dot{Y}_{t=0} = 0. \tag{31c}$$

As we mentioned earlier, the system (31) exhibits a stochastic behaviour which has been studied both numerically and analytically [26–32]. The system is equivalent to one of a two-dimensional particle moving in the potential $U(X,Y) = X^2Y^2$. The configuration space region allowed by the energy conservation can be conventionally divided into several regions with different characters of motion. In the so-called stadium $X \sim Y \lesssim 1$, the motion is almost free, while in four channels along the axes the motion of the particle is strongly affected by the potential. There it can be described by the asymptotic formula [33]

$$Y(t) = -\frac{A}{2}t^2 + W_0 t + Y_0, \tag{32a}$$

$$X(t) = \frac{1}{g^2}\sqrt{\frac{2A}{Y(t)}}\cos\left[g\left(-\frac{A}{6}t^3 + \frac{W_0}{2}t^2 Y_0 t + \varphi_0\right)\right], \tag{32b}$$

where

$$A = \frac{V_0^2 + g^2 X_0^2 Y_0^2}{Y_0}, \qquad \varphi_0 = \arccos\sqrt{\frac{X_0^2 Y_0^2}{V_0^2 + g^2 X_0^2 Y_0^2}}, \tag{32c}$$

and

$$X_0 = X(t_0), \qquad Y_0 = Y(t_0), \tag{32d}$$

$$V_0 = \dot{X}(t_0), \qquad W_0 = \dot{Y}(t_0), \tag{32e}$$

t_0 being the time of entrance into the channel. Here we assumed that the particle is in the region where $|X| \ll |Y|$. (The opposite case is obtained by the interchange of X and Y.)

In the channel the particle reaches the maximal value of $Y \sim W_0^2/A$, after which it is reflected back to the stadium. The instability arises when the particle passes through the stadium and enters a new channel. Generally, therefore, the motion of the particle is stochastic. There is also a discrete set of trajectories which are periodic. Thus, depending on initial conditions the system can move in a regular periodic way, although this motion is unstable as an arbitrary small perturbation can push the system to the stochastic regime.

The asymptotic formulae (32a) and (32b) can provide a reliable description of the system for a certain period of time for extremal cases when the lump centre separation distance is either large ($q \equiv \mathrm{e}^{-|u|^2} \ll 1$) or small ($\sqrt{1-q^2} \ll 1$). Thus, if $u \to \infty$ ($q \ll 1$) then for times less than $t_{\mathrm{stoch}} = gq^{-1}$ one has the asymptotic solution,

$$Y(t) = \frac{q^2\sqrt{1-q^2}}{4g^2}t^2 - \sqrt{1-q^2}, \tag{33a}$$

$$X(t) = \sqrt{\frac{q^2(1-q^2)^{1/2}}{(q^2(1-q^2)^{1/2}/4g^2)t^2 - (1-q^2)^{1/2})}}\cos\left[\frac{1}{g}\left(\frac{q^2\sqrt{1-q^2}}{4g^2}t^3 - \sqrt{1-q^2}t\right)\right]. \tag{33b}$$

In the opposite case when the lumps are close one can again give a reliable description for of the dynamics by the following asymptotic formula:

$$X(t) = -\frac{q(1-q^2)}{4g^2}t^2 + q, \tag{34a}$$

$$Y(t) = \sqrt{\frac{q(1-q^2)}{-(q(1-q^2)/4g^2)t^2 + q}}\cos\left[\frac{1}{g}\left(-\frac{q(1-q^2)}{4g^2}t^3 + qt\right)\right], \tag{34b}$$

valid for times up to the order of $t_{\mathrm{stoch}} = g(1-q^2)^{-1/2}$, after which the system approaches the stadium, where we cannot control its motions.

There is also one particular separation distance which corresponds to periodic motion. This happens for the initial conditions $X|_{t=0} = -Y|_{t=0} = 1/\sqrt{2}$ or $u = \sqrt{\theta \ln 2}$. In this case the motion is periodic and is given by $X(t) = Y(t) \equiv f(t)$, where for $f(t)$ we have the (implicit) formula

$$f(t): \qquad t = \int_{1/\sqrt{2}}^{f} \frac{\mathrm{d}u}{\sqrt{1/4 - u^4}}. \tag{35}$$

Now, let us recall that in terms of $X(t)$ and $Y(t)$ the dynamical field $X_i(t, \bar{z}, z)$ describing the lumps takes according to equation (26) the following form:

$$\begin{aligned} X_1(t, \bar{z}, z) &= \tfrac{1}{2}\sigma_0(\bar{z}, z) + X(t)\sigma_3(\bar{z}, z) + Y(t)\sigma_1(\bar{z}, z) \\ &= \frac{1 - \mathrm{e}^{-\frac{1}{2}|z|^2} X(t) - \sqrt{1 - \mathrm{e}^{-|u|^2}}\, Y(t)}{1 - \mathrm{e}^{-|u|^2}} \mathrm{e}^{-2|z-\frac{u}{2}|^2} \\ &\quad + \frac{1 - \mathrm{e}^{-\frac{1}{2}|z|^2} X(t) + \sqrt{1 - \mathrm{e}^{-|u|^2}}\, Y(t)}{1 - \mathrm{e}^{-|u|^2}} \mathrm{e}^{-2|z+\frac{u}{2}|^2} \\ &\quad + \frac{X(t) - \mathrm{e}^{-\frac{1}{2}|u|^2}}{1 - \mathrm{e}^{-|u|^2}} \mathrm{e}^{-2|z|^2} (\mathrm{e}^{\bar{z}u - z\bar{u}} + \mathrm{e}^{-\bar{z}u + z\bar{u}}), \end{aligned} \tag{36a}$$

and

$$\begin{aligned} X_2(t, \bar{z}, z) &= \tfrac{1}{2}\sigma_0(\bar{z}, z) + X(t)\sigma_3(\bar{z}, z) - Y(t)\sigma_1(\bar{z}, z) \\ &= \frac{1 - \mathrm{e}^{-\frac{1}{2}|z|^2} X(t) + \sqrt{1 - \mathrm{e}^{-|u|^2}}\, Y(t)}{1 - \mathrm{e}^{-|u|^2}} \mathrm{e}^{-2|z-u/2|^2} \\ &\quad + \frac{1 - \mathrm{e}^{-\frac{1}{2}|z|^2} X(t) - \sqrt{1 - \mathrm{e}^{-|u|^2}}\, Y(t)}{1 - \mathrm{e}^{-|u|^2}} \mathrm{e}^{-2|z+u/2|^2} \\ &\quad + \frac{X(t) - \mathrm{e}^{-\frac{1}{2}|u|^2}}{1 - \mathrm{e}^{-|u|^2}} \mathrm{e}^{-2|z|^2} (\mathrm{e}^{\bar{z}u - z\bar{u}} + \mathrm{e}^{-\bar{z}u + z\bar{u}}), \end{aligned} \tag{36b}$$

where the functions $\sigma_{1,3}$ are given by equations (27). Let us note that the function $\sigma_1(\bar{z}, z)$ is localized at the points where the lumps are, i.e. at $z = \pm u/2$, while the function $\sigma_3(\bar{z}, z)$ is also nonzero in the vicinity of the origin, which is the mid-point between the lumps.

The analysis of the solution (36) reveals that once left in their positions the lumps will not tend to move away from them but engage in a stochastic change of their heights as well as creation of a small lump at the mid-point between them. This process can be reliably described for short amounts of time in the limits when the lumps are placed very close together or very far apart, each case degenerating to stochastic motion of the heights of the lumps.

3.3. Exact description: lumps in motion

The difference arising for moving lumps is in the initial values for the velocities. Since a generic initial condition for the velocities can complicate the system, making it infinite dimensional again, we restrict ourselves to such initial data which correspond to the rigid motion of the lumps.

Thus, one has to replace the initial values for the velocities by the following:

$$\dot{X}_i(u)|_{t=0} = \frac{\partial X_i}{\partial u}\dot{u}\bigg|_{t=0} + \frac{\partial X_i}{\partial \bar{u}}\dot{\bar{u}}\bigg|_{t=0}, \tag{37}$$

where $X_i(u)$ is the lump configuration. Explicitly, using (14), one has

$$\dot{X}_1|_{t=0} = -\tfrac{1}{4}(\bar{v}u + \bar{u}v)X_1|_{t=0} + \tfrac{1}{2}(v\bar{a}X_1 + \bar{v}X_1 a)|_{t=0}, \tag{38a}$$

$$\dot{X}_2|_{t=0} = -\tfrac{1}{4}(\bar{v}u + \bar{u}v)X_2|_{t=0} - \tfrac{1}{2}(v\bar{a}X_2 + \bar{v}X_2 a)|_{t=0}, \tag{38b}$$

where $v = \dot{u}(t = 0)$, and solve the infinite-dimensional operator equation (5).

Applying the same strategy as in the case of lumps at rest we see that the initial data are given by operators which are nonzero only in a four-dimensional subspace $\mathcal{H}_u^v$ of the infinite-dimensional Hilbert space $\mathcal{H}$, which is spanned by two old vectors $|\pm u/2\rangle$ and two new vectors $\bar{a}|\pm u/2\rangle = 2(\partial/\partial u|\pm u/2\rangle)$. Let us note that they are all linear independent for $u \neq 0$. Therefore, the system is reduced to the four-dimensional matrix model.

The difference of this case from one with lumps at rest resides only in more complicated technical details, therefore we shall not discuss it here.

The qualitative picture one has in this situation does not change much in comparison with the case of lumps a rest. Just as in the previous case, there is a stochastic dynamics of the heights of the lumps and creation of 'baby lumps', while the centres of the lumps will keep moving with constant velocities. Indeed, for accelerating lump operators X_i are nonzero out of the subspace $\mathcal{H}_u^v$, which, as we know, does not happen.

In general, the solution is given by a linear combination with time-dependent coefficients of functions (51), their first derivatives $(\partial\sigma_\alpha/\partial u)$, $(\partial\sigma_\alpha/\partial\bar{u})$ and some of their second derivatives, such as $(\partial^2\sigma_\alpha/\partial u\partial\bar{u})$.

3.4. The Gauss law

Once we want to relate our system to the the Yang–Mills/BFSS model we have to take care about the Gauss law constraint, which is obtained from the variation of the A_0 component of the original gauge-invariant noncommutative Yang–Mills or BFSS action. This constraint appears as follows:

$$L = [X_i, \dot{X}_i] = 0. \tag{39}$$

As we discussed at the beginning of this section the equations of motion imply that the quantity (39) is at least conserved. Indeed, using the equations of motion one has

$$\dot{L} = \mathrm{i}[X_i, \ddot{X}_i] = 0. \tag{40}$$

Therefore to obtain a self-consistent solution for the Yang–Mills/M(atrix) theory one has to verify that the Gauss law vanishes on initial data, $L|_{t=0} = 0$. For zero-velocity initial conditions this is implied automatically, while for the moving lumps one has

$$L = \mathrm{i}[X_i, \dot{X}_i]|_{t=0} = \mathrm{i}v\left[\left(\frac{\bar{u}}{2} - \bar{a}\right)X_1 + \left(\frac{\bar{u}}{2} + \bar{a}\right)X_2\right] + \mathrm{H.c.}, \tag{41}$$

where 'H.c.' stands for the Hermitian conjugate.

The equation (41) implies that L is zero only when the velocity v vanishes. Therefore moving lumps violate the Gauss law.

As in the ordinary gauge model the violation of the Gauss law for nonzero velocities can be interpreted as the presence of a nontrivial electric charge background. Indeed, in the presence of external sources the Gauss law becomes

$$L' = \mathrm{i}[X_i, \dot{X}_i] + \rho = 0, \tag{42}$$

where ρ is some electric charge density which appears in the action as a term $\Delta S_{\text{charge}} = \int \mathrm{d}^{p+1}x\, \rho X_0$ and which is chosen to cancel (41) exactly. (Here we are not going to analyse in what conditions such charge density can emerge.)

As a result we have that the Gauss law is satisfied automatically in the case of the lumps at rest, while moving lumps generate some background charge distribution.

3.5. More dimensions

One can perform analogous analysis in more than (2 + 1) dimensions.

As most analysis uses the operator formalism, the only difference which appears in $p+1$ dimensions is that one has to compute the Weyl symbols of sigma matrices with respect to a different background, for example one given by (6).

Computation of the $(p+1)$-dimensional analogue of equations (27) yields

$$\sigma_1(x) = \frac{2}{\sqrt{1-\mathrm{e}^{-\frac{1}{2}u\cdot G\cdot u}}}\left(\mathrm{e}^{-(x-u/2)\cdot G\cdot(x-u/2)} - \mathrm{e}^{-(x+u/2)\cdot G\cdot(x+u/2)}\right), \tag{43a}$$

$$\sigma_2(x) = -\frac{4\mathrm{i}\mathrm{e}^{-x\cdot G\cdot x}}{\sqrt{1-\mathrm{e}^{-\frac{1}{2}u\cdot G\cdot u}}}\sin u\times x, \tag{43b}$$

$$\sigma_3(x) = -\frac{2\mathrm{e}^{-u\cdot G\cdot u}}{1-\mathrm{e}^{-\frac{1}{2}u\cdot G\cdot u}}\left(\mathrm{e}^{-(x-u/2)\cdot G\cdot(x-u/2)} + \mathrm{e}^{-(x+u/2)\cdot G\cdot(x+u/2)}\right) + \frac{2\mathrm{e}^{-x\cdot G\cdot x}}{1-\mathrm{e}^{-\frac{1}{2}u\cdot G\cdot u}}\cos u\times x, \tag{43c}$$

$$\mathbb{I}(x) = \sigma_0(x) = \frac{2}{1-\mathrm{e}^{-\frac{1}{2}u\cdot G\cdot u}}\left(\mathrm{e}^{-(x-u/2)\cdot G\cdot(x-u/2)} + \mathrm{e}^{-(x+u/2)\cdot G\cdot(x+u/2)}\right) - \frac{4\mathrm{e}^{-x\cdot G\cdot x-\frac{1}{4}u\cdot G\cdot u}}{1-\mathrm{e}^{-\frac{1}{2}u\cdot G\cdot u}}\cos u\times x, \tag{43d}$$

where we introduced the notation $u\times x = \theta_{\mu\nu}u^\mu x^\nu$, $\mu, \nu = 1, \ldots, p$, and squares are computed with the metric $G = +\sqrt{-\theta^{-2}}$. In the basis for which the noncommutativity matrix $\theta^{\mu\nu}$ takes the canonical form

$$\theta^{\mu\nu} = \begin{pmatrix} \theta_{(1)}\mathrm{i}\sigma_2 & 0 & 0 & \ldots \\ 0 & \theta_{(2)}\mathrm{i}\sigma_2 & 0 & \ldots \\ 0 & 0 & \theta_{(3)}\mathrm{i}\sigma_2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{44}$$

the metric G is diagonal,

$$G_{\mu\nu} = \begin{pmatrix} \theta_{(1)}^{-1}\mathbb{I}_2 & 0 & 0 & \ldots \\ 0 & \theta_{(2)}^{-1}\mathbb{I}_2 & 0 & \ldots \\ 0 & 0 & \theta_{(3)}^{-1}\mathbb{I}_2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{45}$$

where

$$\mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{46}$$

Then the solution is given by an equation similar to (26) but with respective two-dimensional functions replaced by $\mathbb{I}(x)$ and $\sigma_\alpha(x)$.

4. Discussion and conclusions

In this paper we have considered the dynamics of interacting noncommutative lumps.

The naive approach for the dynamics is obtained when one considers the motion of the lumps as rigid structures and does not take into consideration their deformations. The only dynamical parameters in this case are the positions of the lumps. In this approximation the dynamics of the lump pair is described by a cup-shaped potential having minimum at the origin, Gaussian decay at infinity and an unstable equilibrium at the distance $\sqrt{\theta \ln 2}$.

The exact analysis in the framework of the original noncommutative theory, however, refutes the result of the above approximation. It appears that in fact it is the shape which is mostly affected by the the interaction, and not the motion of the centres of lumps.

Another interesting observation is that the problem of the noncommutative Yang–Mills model is reduced to one in a finite-dimensional matrix model. Thus, in the case of two lumps starting with zero velocities, the exact description reduces to a 2×2 matrix model. In particular we have that the $U(1)$ part of this model has trivial dynamics, while the remaining $SU(2)$ part generally exhibits stochastic behaviour.

The property of this dynamics that it does not affect the motion along the line connecting the lumps appears to be anti-intuitive to what one could expect from interaction of (quasi)particle objects. Let us note that an analogous situation can be met in vortex dynamics in solid-state physics [34–36].

The results of this work can be easily generalized to the case of lumps with arbitrary mutual Hilbert space polarizations not related to shifts along noncommutative space. The dynamics of such lumps or branes does not differ qualitatively from the shifted ones, but in this case the simple physical interpretation is missing. However, from the point of view of mathematical completeness it would be worth considering, and this will probably be done in future research.

It seems that the interpretation in terms of branes when the heights of the lumps have the meaning of coordinates of the zero-brane in the direction transversal to the noncommutative brane is the most natural. (In this picture the Yang–Mills component of the model describes the dynamics of longitudinal degrees of freedom of the noncommutative brane, while the scalar fields describe the transversal ones.) Since lumps form localized configurations they can be attributed the sense of zero-branes. In this context it appears that the dynamics of interacting zero-branes affects only the motion in the transversal directions in which D0-branes are stochastically 'bouncing'.

We have also learnt that the dynamics of two interacting zero-branes is described by the $U(2)$ M(atrix) model in the case when the branes do not move or do not change their polarizations. If the branes are in motion one needs a matrix model of higher dimension to describe it. This is slightly in contrast to the interpretation of the finite N matrix model, which is believed to describe exactly N branes.

So far we have considered only one pair of interacting lumps. It would also be of interest to extend the analysis of this paper to a larger number of lumps, and eventually to consider a gas of lumps.

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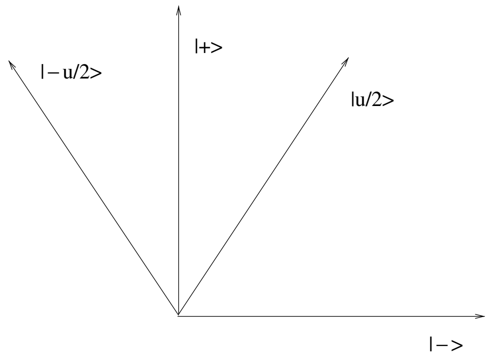

Figure A.1. The orthogonal vectors $|\pm\rangle$ constructed from unimodular but nonorthogonal $|\pm u/2\rangle$.

useful discussions at the QFT Department of Steklov Mathematical Institute in Moscow. Discussion with I Ya Aref'eva and P B Medvedev improved my understanding the dynamics of the $SU(2)$ matrix model. This work was supported by RFBR grant no 99-01-00190, INTAS grants no 01-262 and no 99 0590, Scientific School support grant no 00-15-96046 and a NATO fellowship programme.

Appendix. Useful formula connecting the two-dimensional representation with other representations

Here we summarize the formula connecting the three main representations of the objects used in this paper, Hilbert space operator, noncommutative functions (Weyl symbols) and two-dimensional matrices.

The two-dimensional space $\mathcal{H}_u$ for $u \neq 0$ is the span of the two vectors $|-u/2\rangle = \mathrm{e}^{-\frac{1}{8}|u|^2}\mathrm{e}^{-\frac{1}{2}\bar{a}u}|0\rangle$ and $|u/2\rangle = \mathrm{e}^{-\frac{1}{8}|u|^2}\mathrm{e}^{\frac{1}{2}\bar{a}u}|0\rangle$, where $|0\rangle$ is the oscillator vacuum state.

The vectors $|\pm u/2\rangle$ have unit magnitudes but are not orthogonal. One can easily construct an orthonormal basis consisting of vectors $\{|+\rangle, |-\rangle\}$ given by equations (20a) and (20b) (see figure A.1).

An arbitrary Hermitian operator acting in this two-dimensional subspace can be expanded in terms of ordinary Pauli matrices and the unit matrix,

$$\sigma_0 \equiv \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{47}$$

as follows:

$$X = X^0 + X^\alpha \sigma_\alpha, \tag{48}$$

where X^α are computed as

$$X^\alpha = \tfrac{1}{2}\,\mathrm{tr}\, X\sigma_\alpha. \tag{49}$$

On the other hand, as operators over the Hilbert space the two-dimensional unit matrix[5] and Pauli matrices can be expressed as noncommutative functions through their Weyl symbols.

[5] Which is the projector to the two-dimensional subspace $\mathcal{H}_u$ of the Hilbert space.

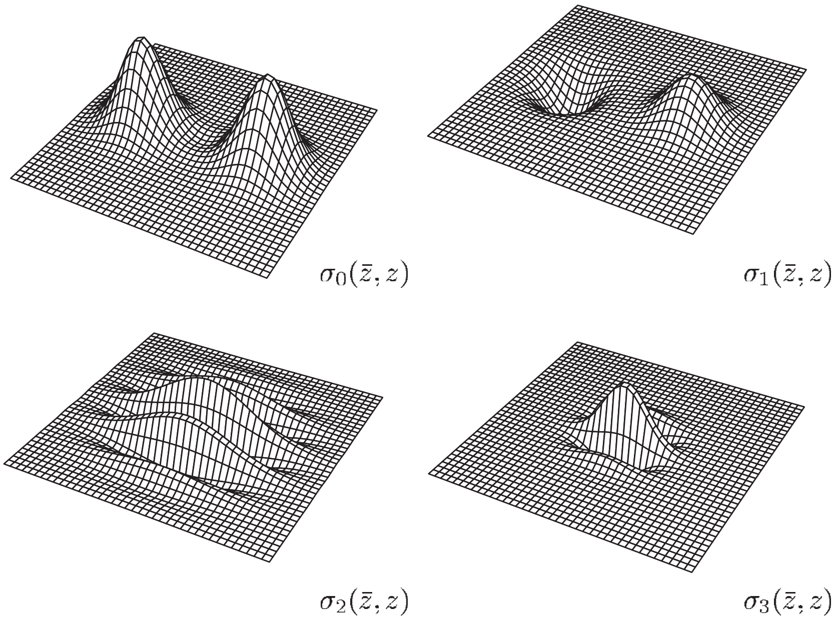

Figure A.2. Plots of the profiles of the functions $\sigma_{0,1,2,3}(\bar{z}, z)$.

The Weyl symbols of operators with bounded square trace to which undoubtedly belong $\mathbb{I}_2$ and σ_α can be found by a direct formula,

$$X \sim \int \mathrm{d}\bar{k}\, \mathrm{d}k\, \mathrm{e}^{\mathrm{i}(\bar{k}z + k\bar{z})}\, \mathrm{tr}\, X \mathrm{e}^{-\mathrm{i}(\bar{k}a + k\bar{a})}. \tag{50}$$

Technically, one can write the matrices in the (nonorthogonal) basis of $|\pm u/2\rangle$ and use the Weyl symbols for the following operators:

$$\begin{aligned}
|u/2\rangle\langle u/2| &\sim 2\mathrm{e}^{-2|z-u/2|^2}, \\
|-u/2\rangle\langle -u/2| &\sim 2\mathrm{e}^{-2|z+u/2|^2}, \\
|u/2\rangle\langle -u/2| &\sim 2\mathrm{e}^{-2|z|^2 + (\bar{z}u - z\bar{u})}, \\
|-u/2\rangle\langle u/2| &\sim 2\mathrm{e}^{-2|z|^2 - (\bar{z}u - z\bar{u})},
\end{aligned} \tag{51}$$

which can be easily computed.

The sigma matrices are expressed in the nonorthogonal basis of $|\pm u/2\rangle$ as follows:

$$\sigma_1 = \frac{1}{\sqrt{1 - \mathrm{e}^{-|u|^2}}}(|u/2\rangle\langle u/2| - |-u/2\rangle\langle -u/2|), \tag{52a}$$

$$\sigma_2 = \frac{1}{\sqrt{1 - \mathrm{e}^{-|u|^2}}}(|u/2\rangle\langle -u/2| - |-u/2\rangle\langle u/2|), \tag{52b}$$

$$\begin{aligned}
\sigma_3 = &-\frac{\mathrm{e}^{-\frac{1}{2}|u|^2}}{1 - \mathrm{e}^{-|u|^2}}(|u/2\rangle\langle u/2| + |-u/2\rangle\langle -u/2|) \\
&+ \frac{1}{1 - \mathrm{e}^{-|u|^2}}(|u/2\rangle\langle -u/2| + |-u/2\rangle\langle u/2|),
\end{aligned} \tag{52c}$$

and, finally

$$\sigma_0 = \frac{1}{1-e^{-|u|^2}}(|u/2\rangle\langle u/2| + |-u/2\rangle\langle -u/2|) + \frac{e^{-\frac{1}{2}|u|^2}}{1-e^{-|u|^2}}(|u/2\rangle\langle -u/2| + |-u/2\rangle\langle u/2|). \tag{52d}$$

Inserting (51) into (52) one finds immediately the functions (27) of the third section. The plots of functions $\sigma_\alpha(\bar{z}, z)$ can be seen in figure A.2.

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